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Wave motion in a viscous fluid of variable depth Part 2. Moving contact line

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An earlier derivation (Miles 1990*a*) of the partial differential equation for the complex amplitude of a gravity–capillary wave in a shallow, viscous liquid of variable depth and fixed contact line is extended to accommodate a meniscus with a moving contact line at which the slope of the meniscus is assumed to be proportional to (but not necessarily in phase with) the velocity. The motion of the contact line implies capillary dissipation, which is absent for a fixed contact line. The results are applied to the normal reflection of a wave incident from a region of uniform depth on a beach of uniform slope. The reflection coefficient has the form $R = R_1 R_\nu R_c$, where R_1 is the coefficient for an ideal fluid, and R_ν and R_c comprise the respective effects of viscosity and capillarity.

1. Introduction

I present here the extension of an earlier analysis (Miles 1990*a*, hereinafter referred to as I) of linear wave motion in a viscous fluid of variable depth and fixed contact line to accommodate a meniscus with a moving contact line. I assume, as in I, that

$$Kh \ll 1, \quad K\delta_* \ll \sigma^2 \ll 1, \quad \sigma l_c = O(\delta), \tag{1.1 a–c}$$

where

$$\frac{1}{K} \equiv \frac{g}{\omega^2}, \quad \delta \equiv \left(\frac{\nu}{\omega}\right)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} \equiv (1+i)\,\delta_*, \quad l_c \equiv \left(\frac{T}{\rho g}\right)^{\frac{1}{2}} \tag{1.2 a–c}$$

are inertial, viscous and capillary lengthscales, ω is the angular frequency of the wave motion, ν is the kinematic viscosity, T is the surface tension, h is the depth, and σ is the slope of the beach at the straight shoreline, where $h \sim \sigma x$. I also assume that the static contact angle is prescribed and that the dynamical variation of the contact angle is linearly related to the contact-line velocity according to (cf. Hocking 1987)

$$Z_x = sZ_t, \tag{1.3}$$

where Z is the complex amplitude of the dynamical component of the free-surface displacement, and s is the phenomenological parameter that has the dimensions of inverse velocity and is expected to be a complex function of frequency.

It is worth emphasizing that (1.3) is the most general linear boundary condition that is admissible for the conventional (Laplace) model of capillary action. The parameter s, which appears to have been measured only for steady motion, could be determined for harmonic motion from measurements of the frequency and damping of standing waves in a vertical cylinder, but this would require a rather precise determination of the corresponding viscous effects.

The invocation of (1.3) in place of the assumption of a fixed contact line, together with the requirement that the shear stress be bounded, requires the conventional noslip condition on the bottom to be replaced by a slip condition that I pose in the form (see Dussan V. 1979 for discussion and references)

$$\boldsymbol{U} = l_{\mathrm{b}}(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{U} \quad (z=-h), \tag{1.4}$$

where $\boldsymbol{U}$ is the complex amplitude of the tangential velocity, $\boldsymbol{n}$ is the normal directed into the fluid, and l_{b} is a slip length that (by hypothesis) decays away from the contact line and, like s, may be a complex function of the frequency. Physical considerations imply that l_{b} is significant only in a small neighbourhood of the contact line, and it does not enter the outer (asymptotic) solution; see e.g. §4.

The boundary condition on the tangential stress at the free surface may be posed in the comparable form

$$\boldsymbol{U} = -l_{\mathrm{f}}(\boldsymbol{U}_z + \boldsymbol{\nabla}_x W), \tag{1.5}$$

where $\boldsymbol{x}$ and z are the horizontal and vertical coordinates (z is positive up), $\boldsymbol{U}$ and W are the complex amplitudes of the horizontal and vertical components of the velocity, $\boldsymbol{\nabla}_x$ is the horizontal component of $\boldsymbol{\nabla}$, and l_{f} is a length that may be expressed in terms of the surface-film parameters (Miles 1967, 1990*c*) or, more conveniently, regarded as a phenomenological parameter to be determined from laboratory measurements. The special cases of vanishing tangential stress or vanishing tangential velocity (inextensible film) are obtained by letting $l_{\mathrm{f}}/\delta \rightarrow \infty$ or 0.

The primary result in I is the partial differential equation

$$\boldsymbol{\nabla}\cdot(H\boldsymbol{\nabla}\mathscr{L}Z)+KZ = 0, \tag{1.6}$$

where $H = H(h/\delta)$ and $\mathscr{L} = 1-l_{\mathrm{c}}^2\nabla^2$. In §3 below, following the determination of the meniscus $z=z_0(x)$ in §2, I extend the formulation of I by replacing the surface condition $\boldsymbol{U}_z+\boldsymbol{\nabla}_x W=0$ by (1.5) and the bottom condition $\boldsymbol{U}=0$ by (1.4) and obtain (1.6) with

$$H = H\left(\frac{h+z_0}{\delta};\frac{l_{\mathrm{b}}}{\delta},\frac{l_{\mathrm{f}}}{\delta}\right), \quad \mathscr{L}Z = Z - l_{\mathrm{c}}^2\boldsymbol{\nabla}\cdot(C\boldsymbol{\nabla}Z), \tag{1.7a, b}$$

where H and C are given by (3.6) and (2.5*b*). The behaviour of H for $h+z_0 \gg |\delta|$ is similar to that in I and corresponds to a conventional boundary-layer approximation. But, whereas (in I) $H = O(h/\delta)^3$ as $h/\delta \rightarrow 0$ with $l_{\mathrm{b}}=0$ and either $l_{\mathrm{f}}=0$ or $l_{\mathrm{f}}=\infty$, $\mathrm{H} = O[(h+z_0)^2/\delta^2]$ as $(h+z_0)/\delta\rightarrow 0$ with l_{b} bounded away from zero. The singularity of (1.6) at the contact line, $h+z_0=0$, is regular in each of these limits, but the exponents in I are (2, 1, 0, 0), in consequence of which the contact line must be fixed in order to avoid a singularity in the shear stress; the exponents in the present case are (2, 1, 1, 0), and the shear stress is bounded for a moving contact line.

In §4, I solve (1.6), subject to the contact-line condition (1.3), for the normal reflection of a plane wave by a laboratory beach ($h=\sigma x$ for $x \leqslant x_1$ and $h=h_1$ for $x \geqslant x_1$), thereby generalizing the result for a fixed contact line (Miles 1990*b*). The magnitude of the reflection coefficient has the form

$$|R| = \exp(-\rho_\nu - \rho_{\mathrm{c}}), \tag{1.8}$$

where

$$\rho_\nu = \frac{2\pi K}{\sigma^2}\,\mathrm{Im}\left[\delta\left(\frac{l_{\mathrm{f}}+2\delta}{l_{\mathrm{f}}+\delta}\right)\right], \quad \rho_{\mathrm{c}} = \frac{2\pi K l_{\mathrm{c}}}{\sigma}\,\mathrm{Im}\left(\frac{1}{1-\mathrm{i}s\omega l_{\mathrm{c}}}\right) \tag{1.9a, b}$$

represent viscous and capillary dissipation (after neglecting a weak dependence of ρ_{c} on the contact angle; see §4). The extrapolation (dubious at best) of Ablett's (1923) data for s yields the rough estimate of $\rho_{\mathrm{c}} \approx 10^{-2}\sigma^{-1}T^{-1}$ for a wave of period T seconds on clean water. This yields $\rho_{\mathrm{c}} = 10^{-1}$ compared with $\rho_\nu = 10^0$ for $\sigma = 10^{-1}$ and $T =$ 1 s.

I also have generalized the solution of I §6 for a Stokes edge wave and find that the logarithmic decrement is given by the exponent $\rho_\nu + \rho_c$ in (1.8).

2. The meniscus

The static free surface, $z = z_0(x)$, is governed by the capillary equation

$$z_0 = \frac{T}{\rho g R_0} = \frac{l_c^2 z_0''}{(1+z_0'^2)^{\frac{3}{2}}}, \tag{2.1}$$

where R_0 is the radius of curvature. We assume a uniformly sloping bottom, $z = -h = -\sigma x$, in that domain in which z_0' differs significantly from zero. The contact line, $x = x_c$, is located by

$$\tan^{-1} z_0' + \sigma = \theta_0, \quad z_0 = -\sigma x_c \equiv z_c \quad (x = x_c), \tag{2.2a, b}$$

wherein we have approximated $\tan^{-1}\sigma$ by σ and θ_0 is the static contact angle. The solution of (2.1), (2.2), and the requirement $z_0 \to 0$ as $x \to \infty$, is given by (a particular case of Euler's *elastica*; cf. Lamb 1928, §127)

$$\frac{x - x_c}{l_c} = \log\left(\frac{\tan\frac{1}{4}\psi_c}{\tan\frac{1}{4}\psi}\right) + 2(\cos\tfrac{1}{2}\psi_c - \cos\tfrac{1}{2}\psi), \tag{2.3a}$$

$$\frac{z_0}{l_c} = -2\sin\tfrac{1}{2}\psi, \quad z_0'(x) = \tan\psi, \tag{2.3b, c}$$

where

$$\psi_c = \theta_0 - \sigma, \quad x_c = 2\sigma^{-1} l_c \sin\tfrac{1}{2}\psi_c, \tag{2.4a, b}$$

and ψ varies monotonically from 0 at $x = \infty$ to ψ_c at the contact line.

The contact angle θ_0 lies in $(0, \pi)$, with $\psi_c \lesseqgtr 0, z_c \gtreqless 0$ and $x_c \lesseqgtr 0$ for $\theta_0 \lesseqgtr \sigma$. We exclude those solutions that contain closed loops, which presumably are unstable, or intersect the bottom in $x > x_c$.

The vertical component of the capillary force induced by the dynamical perturbation $z - z_0 = \zeta$ is given by

$$T\left(\frac{1}{R} - \frac{1}{R_0}\right) = T\nabla\cdot(C\nabla\zeta), \quad C = (1+z_0'^2)^{-\frac{3}{2}} = \cos^3\psi. \tag{2.5a, b}$$

3. The shallow-water equation

The continuity and linearized Navier–Stokes equations are (Lamb 1932, §328)

$$\nabla\cdot\boldsymbol{q} = 0, \quad \boldsymbol{q}_t = -\nabla\left(\frac{p}{\rho} + gz\right) + \nu\nabla^2\boldsymbol{q} \quad (-h < z < z_0 + \zeta), \tag{3.1a, b}$$

where $\boldsymbol{q} = (\boldsymbol{u}, w)$ is the velocity, $\boldsymbol{u}$ and w are the horizontal and vertical components thereof, p is the total pressure, gz is the gravitational potential, ν is the kinematic viscosity, and ζ is the free-surface displacement. The linearized free-surface conditions, projected on $z = z_0$, are (1.5) and

$$w = \zeta_t + \boldsymbol{u}\cdot\nabla z_0, \quad -\frac{p_d}{\rho} + g\zeta + 2\nu w_z = \frac{T}{\rho}\nabla\cdot(C\nabla\zeta) \quad (z = z_0), \tag{3.2a, b}$$

where p_d is the hydrodynamic pressure, and l_f is the surface-film parameter. The bottom conditions are (1.4) and

$$w + \boldsymbol{u}\cdot\nabla h = 0 \quad (z = -h). \tag{3.3}$$

Introducing the complex scalar and vector potentials Φ and $\boldsymbol{A}$ according to

$$\left(\boldsymbol{q}, \frac{p_{\mathrm{d}}}{\rho}, \zeta\right) = \operatorname{Re}\{(\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\times\boldsymbol{A}, \mathrm{i}\omega\Phi, Z)\,\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{3.4}$$

proceeding as in I §2, and invoking Kh and $|\boldsymbol{\nabla}h| \ll 1$, we obtain (Appendix A; note that z has been eliminated and that $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}_x$ are now equivalent)

$$\boldsymbol{\nabla}\cdot(H\boldsymbol{\nabla}\mathscr{L}Z) + KZ = 0, \quad \mathscr{L}Z \equiv Z - l_{\mathrm{c}}^2\boldsymbol{\nabla}\cdot(C\boldsymbol{\nabla}Z), \tag{3.5a, b}$$

where

$$H = h + z_0 - \delta\left[\frac{2(\cosh \hbar - 1) + (\lambda_{\mathrm{b}} + \lambda_{\mathrm{f}})\sinh \hbar}{(1 + \lambda_{\mathrm{b}}\lambda_{\mathrm{f}})\sinh \hbar + (\lambda_{\mathrm{b}} + \lambda_{\mathrm{f}})\cosh \hbar}\right], \tag{3.6}$$

$$\hbar \equiv \frac{h + z_0}{\delta}, \quad \lambda_{\mathrm{b}} \equiv \frac{l_{\mathrm{b}}}{\delta}, \quad \lambda_{\mathrm{f}} \equiv \frac{l_{\mathrm{f}}}{\delta}. \tag{3.7a–c}$$

The limiting approximations to H on the assumption that $|\lambda_{\mathrm{b}}| \ll 1$ are

$$H \to \left(\frac{l_{\mathrm{b}}l_{\mathrm{f}}}{l_{\mathrm{b}} + l_{\mathrm{f}}}\right)\hbar^2 \quad (\hbar \to 0) \tag{3.8}$$

and

$$H \sim h - \delta\left(\frac{l_{\mathrm{f}} + 2\delta}{l_{\mathrm{f}} + \delta}\right) \equiv \delta(\hbar - \hbar_*) \quad (\hbar \to \infty). \tag{3.9}$$

We assume that $\mathrm{d}\hbar/\mathrm{d}x$, l_{b} and l_{f} tend to constants as $\hbar \downarrow 0$ and that Z is either independent of, or periodic in, y; (3.5a) then reduces to a fourth-order, ordinary differential equation with a regular singularity (at $\hbar = 0$) of exponents 2, 1, 1, 0 and admits four, linearly independent solutions through the method of Frobenius (Ince 1944, §§16.1–16.3). The boundary conditions at $\hbar = 0$, as determined by (1.3) and the requirement that the mass flux (see (A 8) in Appendix A) vanish at the shoreline, are

$$Z_x + \mathrm{i}\omega s Z = 0, \quad H(\mathscr{L}Z)_x = 0 \quad (\hbar = 0). \tag{3.10a, b}$$

It can be shown that the complex amplitude of the shear stress, $\rho\nu(\boldsymbol{u}_z + \boldsymbol{\nabla}_x w)$, behaves like $\hbar\boldsymbol{\nabla}\mathscr{L}Z$, which is bounded as $\hbar \downarrow 0$. But if $l_{\mathrm{b}} = 0$ (as in I) H vanishes like $\hbar^3$, rather than $\hbar^2$, the exponents of the singularity are 2, 1, 0, 0, and the shear stress then is bounded at $\hbar = 0$ only if Z vanishes there.

4. Reflection from uniform slope

We seek the solution of (3.5) for

$$h = \frac{\sigma x}{h_1} \quad \left(x \leqslant x_1 \equiv \frac{h_1}{\sigma} \gg \frac{\sigma}{K}, \frac{|\delta|}{\sigma}\right) \tag{4.1}$$

on the assumption that the solution in the domain of uniform depth has the form

$$Z = \left(\frac{Z_1}{1+R}\right)[\mathrm{e}^{-\mathrm{i}k_1(x-x_1)} + R\,\mathrm{e}^{\mathrm{i}k_1(x-x_1)}], \quad k_1 = \left(\frac{K}{H_1}\right)^{\frac{1}{2}} \quad (x \geqslant x_1), \tag{4.2a, b}$$

where Z_1 is the complex amplitude of the wave at the toe of the beach ($x = x_1$), and R, which is to be determined, is the reflection coefficient referred to $x = x_1$. Note that H_1, as given by (3.9) for $x = x_1$, has a negative imaginary part, and hence that k_1 has a positive imaginary part, in consequence of the viscous damping in the boundary layers.

We proceed on the assumptions that $K|\delta| \ll \sigma^2 \ll Kh$ and $\sigma l_c = O(\delta)$. Capillarity then is negligible in $h \gg |\delta|$, and (3.5) admits the outer solution

$$Z = Z_1\left[\frac{J_0(\chi)\cos\chi_0 - Y_0(\chi)\sin\chi_0}{J_0(\chi_1)\cos\chi_0 - Y_0(\chi_1)\sin\chi_0}\right], \quad \chi = \frac{2}{\sigma}(KH)^{\frac{1}{2}} \quad \left(\frac{|\delta|}{\sigma} \ll x \leqslant x_1\right), \quad (4.3a, b)$$

where J_0 and Y_0 are Bessel functions, H is given by (3.9),

$$\chi_1 \equiv 2\sigma^{-1}(KH_1)^{\frac{1}{2}} = 2\sigma^{-1}[K(h_1 - \delta \hbar_*)]^{\frac{1}{2}}, \quad (4.3c)$$

and χ_0 is to be determined. $Z(4.2a) = Z(4.3a)$ at $x = x_1$ by construction; requiring $\mathrm{d}Z/\mathrm{d}x$ to be continuous and invoking $Kh_1 \gg \sigma^2$ ($|\chi_1| \gg 1$), we obtain

$$R = \exp\left\{2\mathrm{i}\tan^{-1}\left[\frac{J_1(\chi_1)\cos\chi_0 - Y_1(\chi_1)\sin\chi_0}{J_0(\chi_1)\cos\chi_0 - Y_0(\chi_1)\sin\chi_0}\right]\right\} \quad (4.4a)$$

$$= \exp\{2\mathrm{i}(\chi_0 + \chi_1 - \tfrac{1}{4}\pi)\}[1 + O(\chi_1^{-1})]. \quad (4.4b)$$

Turning to the solution of (3.5) in $h = O(\delta)$, we introduce

$$\alpha \equiv \frac{Kl_c^2}{\delta}, \quad \beta \equiv l_c s\omega, \quad \gamma \equiv \frac{\sigma l_c}{\delta}, \quad \xi \equiv \frac{x - x_c}{l_c}, \quad H_\delta(\xi) \equiv \frac{H}{\delta}, \quad (4.5a\text{–}e)$$

and transform (3.5) and (3.10) to

$$[H_\delta(\mathscr{L}Z)']' + \alpha Z = 0, \quad \mathscr{L}Z = 1 - (CZ')', \quad (4.6a, b)$$

and
$$Z' + \mathrm{i}\beta Z = 0, \quad H_\delta(\mathscr{L}Z)' = 0 \quad (\xi = 0), \quad (4.7a, b)$$

where $Z = Z(\xi)$. We pose the solution of (4.6) and (4.7) for $|\alpha| \ll 1$ in the form

$$Z = Z^{(0)} + \alpha\hat{Z}, \quad (4.8)$$

where
$$Z^{(0)} = a + bE(\xi), \quad a = Z_0\left(1 + \mathrm{i}\beta\frac{E_0}{E_0'}\right), \quad b = \frac{-\mathrm{i}\beta Z_0}{E_0'}, \quad (4.9a\text{–}c)$$

$Z_0 \equiv Z(0)$, $E_0 \equiv E(0)$, $E_0' \equiv E'(0)$, E is determined by (recall that $C \sim 1$ as $\xi \uparrow \infty$)

$$\mathscr{L}E = 0, \quad E = O(\mathrm{e}^{-\xi}) \quad (\xi \uparrow \infty), \quad \int_0^\infty E\,\mathrm{d}\xi = 1, \quad (4.10a\text{–}c)$$

and $\hat{Z}$ is determined by (note that $(\mathscr{L}Z^{(0)})' = 0$)

$$[H_\delta(\mathscr{L}\hat{Z})']' = -Z, \quad \hat{Z} = \hat{Z}' = 0 \quad (\xi = 0), \quad (4.11a, b)$$

(4.7b), and the matching of (4.8) to (4.3a).

Integrating (4.11a) from 0 to ξ, invoking (4.7b), dividing by H_δ, integrating again, and choosing the constant of integration such that the forcing function for $\mathscr{L}\hat{Z}$ is orthogonal to the eigensolution E, we obtain

$$\mathscr{L}\hat{Z} = M(\xi) - \int_0^\infty EM\,\mathrm{d}\xi \equiv \hat{M}(\xi), \quad M = -\int_{\xi_1}^{\xi}\frac{\mathrm{d}\eta}{H_\delta(\eta)}\int_0^\eta Z(\zeta)\,\mathrm{d}\zeta, \quad (4.12a, b)$$

where ξ_1 is an arbitrary constant of which (4.12a) is independent. (We note that, since $H_\delta = O(\xi^2)$ and $Z = O(1)$ as $\xi \to 0$, $M = O(\log\xi)$, $Z'' = O(\log\xi)$, and $\xi Z''' = O(1)$, so that the shear stress (see last paragraph in §3) is bounded at the contact line.) Integrating (4.12a) by variation of parameters and invoking (4.11b), we obtain

$$\hat{Z}(\xi) = \int_0^\xi [E(\xi)F(\eta) - E(\eta)F(\xi)]\hat{M}(\eta)\,\mathrm{d}\eta, \quad F(\xi) = E(\xi)\int_0^\xi [C(\eta)E^2(\eta)]^{-1}\,\mathrm{d}\eta. \quad (4.13a, b)$$

The sequence (4.8), (4.12) and (4.13) may be solved by iteration, starting from the first approximation $Z = Z^{(0)}$. It follows from $Z = O(1)$ and $H_\delta = O(\xi)$ as $\xi \uparrow \infty$ that $\hat{M} \sim C_1\xi + C_0 \log \xi$ (C_1 and C_0 are constants), and hence from (4.12) that (since $\mathscr{L}\hat{Z} \sim \hat{Z} - Z''$) $\hat{Z} \sim \hat{M}$, by virtue of which the outer approximation to Z at any stage of the iteration is given by

$$Z \sim a + \alpha\hat{M}(\xi), \tag{4.14}$$

which must be matched to (4.3a).

We proceed to the second approximation. Substituting (4.9a) into (4.12b), choosing ξ_1 large enough to justify the asymptotic approximation (3.9) or, after invoking (4.5c, d) and (2.4b)

$$H_\delta \sim \gamma(\xi - \xi_*), \quad \xi_* = \frac{\hbar_*}{\gamma} - \frac{x_c}{l_c}, \tag{4.15a, b}$$

and remarking that

$$\int_0^\xi Z^{(0)}(\eta)\,d\eta \sim a\xi + b + O(e^{-\xi}), \tag{4.16}$$

by virtue of (4.10a, c), we obtain

$$M \sim -\gamma^{-1}\left[a(\xi - \xi_1) + (a\xi_* + b)\log\left(\frac{\xi - \xi_*}{\xi_1 - \xi_*}\right)\right] + O(\alpha), \tag{4.17}$$

wherein $O(\alpha)$ may comprise $O(\alpha \ln \alpha)$. Substituting (4.17) into (4.14), we obtain

$$Z \sim a\left[1 - \frac{\alpha}{\gamma^2}H_\delta - \frac{\alpha}{\gamma}\left(\frac{\hbar_*}{\gamma} - \frac{x_c}{l_c} + \frac{b}{a}\right)(\log H_\delta + L) + O(\alpha^2)\right], \tag{4.18}$$

where L is a constant that could be determined through a more complete determination of M than that provided by (4.17) but is more readily determined from the following matching.

Returning to the outer approximation (4.3), letting $H_\delta \to 0$ therein, and invoking $K/\sigma^2 = \alpha/\gamma^2$, we obtain

$$Z \sim A\left[\left(1 - \frac{\alpha}{\gamma^2}H_\delta\right)\cos\chi_0 - \left(\frac{\sin\chi_0}{\pi}\right)\log\left(\frac{C^2\alpha}{\gamma^2}H_\delta\right)\right] \tag{4.19a}$$

where

$$A = Z_1[J_0(\chi_1)\cos\chi_0 - Y_0(\chi_1)\sin\chi_0]^{-1} \tag{4.19b}$$

and $C = 1.78\ldots$ is Euler's constant. Matching (4.18) and (4.19), we obtain

$$a = A\cos\chi_0, \quad \tan\chi_0 = \frac{\pi\alpha}{\gamma}\left(\frac{\hbar_*}{\gamma} - \frac{x_c}{l_c} + \frac{b}{a}\right), \quad L = \log\left(\frac{C^2\alpha}{\gamma^2}\right). \tag{4.20a–c}$$

Combining $\chi_0 \approx \tan\chi_0$ with χ_1 (4.3c) in (4.4b), remarking that χ_1 then may be approximated by $2(Kh_1)^{\frac{1}{2}}$ within the existing error factor of $1 + O(1/\chi_1)$, and invoking (4.9b, c), we place the reflection coefficient in the form

$$R = R_1 R_\nu R_c, \tag{4.21a}$$

where

$$R_1 = \exp\left[4i\sigma^{-1}(Kh_1)^{\frac{1}{2}} - \tfrac{1}{2}i\pi\right] \tag{4.21b}$$

comprises the phase shift over the beach ($R = R_1$ for $K\delta = Kl_c = 0$);

$$R_\nu = \exp(2i\pi\sigma^{-2}K\delta\hbar_*) \tag{4.21c}$$

and
$$R_c = \exp\left[-\frac{2\mathrm{i}\pi K}{\sigma}\left(x_c + \frac{\mathrm{i}\beta l_c}{E_0' + \mathrm{i}E_0\beta}\right)\right] \tag{4.21d}$$

comprise the respective effects of viscosity and capillarity and are independent of both h_1 and (as anticipated) l_b. The magnitude of R is given by

$$|R| = |R_\nu R_c| \equiv \exp(-\rho_\nu - \rho_c), \tag{4.22a}$$

where
$$\rho_\nu = \left(\frac{2\pi K}{\sigma^2}\right)\mathrm{Im}\,(\delta \hbar_*), \quad \rho_c = \left(\frac{2\pi K l_c}{\sigma}\right)\left(\frac{-E_0'\beta_r}{|E_0' + \mathrm{i}E_0\beta|^2}\right), \tag{4.22b, c}$$

in which ρ_ν and ρ_c represent viscous and contact-line damping, β_r is the real part of β, and an error factor $1 + O(1/\chi_1)$ is implicit. The exponent ρ_ν reduces to that obtained previously (Miles 1990*b*) for either $\hbar_* = 1(l_f/\delta = \infty)$ or $\hbar_* = 2(l_f/\delta = 0)$.

The approximations

$$E_0 = 1 + \tfrac{1}{4}q_0, \quad E_0' = -(1+q_0), \quad q_0 \equiv 1 - \cos^3\psi_c \tag{4.23a–c}$$

are derived in Appendix B; however, absent reliable estimates of β, it suffices for qualitative estimates of contact-line dissipation to adopt the approximations $E_0 = 1$ and $E_0' = -1$ (which is tantamount to neglecting the meniscus), thereby reducing (4.22*c*) to

$$\rho_c = \left(\frac{2\pi K l_c}{\sigma}\right)\mathrm{Im}\left(\frac{1}{1-\mathrm{i}\beta}\right)[1 + O(q_0)]. \tag{4.24}$$

Ablett's (1923) measurements for *steady* ($\omega \to 0$) flow of water over wax yield $s = 3.7$ s/cm, the extrapolation of which to representative laboratory frequencies implies $\beta = 6.5/T$ for a wave of period T in clean water; ρ_c then may be approximated by $2\pi\omega/\sigma g s = 10^{-2}(\sigma T)^{-1}$, although the extrapolation to $|\beta| \gg 1$ is dubious at best. This estimate suggests that contact-line motion of clean water on a 10% slope could reduce $|R|$ (relative to its value for viscous action alone) by 10% for a one-second wave. On the other hand, the experimental observations of Mahony & Pritchard (1980) for waves of period somewhat less than one second suggest that the contact line remains fixed ($s \to \infty$) for such short periods, at least for small amplitudes.

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Appendix A. Derivation of (3.5)

The derivation of (3.5) from (3.1)–(3.4) follows I §2 after allowance for differences in the boundary conditions. Substituting (3.4) into (3.1) and invoking $\nu \equiv -\mathrm{i}\omega\delta^2$, we obtain

$$\nabla^2\Phi = 0, \quad \delta^2\nabla^2\boldsymbol{A} = \boldsymbol{A}. \tag{A 1a, b}$$

Eliminating $\tilde{p}_d$ from (3.2*b*) with the aid of (3.4), invoking (2.5), introducing $\boldsymbol{U}$ and W, the complex amplitudes of $\boldsymbol{u}$ and w (cf. (3.4)), and the operator $\mathscr{L}$ (3.5*b*), and invoking $T \equiv \rho g l_c^2$ and $\nu = -\mathrm{i}\omega\delta^2$, we transform (3.2) and (3.3) to

$$W = -\mathrm{i}\omega Z + \boldsymbol{U}\cdot\nabla Z_0, \quad \Phi + 2\delta^2 W_z = (g/\mathrm{i}\omega)\mathscr{L}Z, \quad \boldsymbol{U} = -l_f(\boldsymbol{U}_z + \nabla_x W) \quad (z = z_0), \tag{A 2a–c}$$

and $$W + \boldsymbol{U}\cdot\nabla h = 0, \quad \boldsymbol{U} = l_{\mathrm{b}}\, \boldsymbol{U}_z\,(z = -h). \tag{A 3a, b}$$

We pose the shallow-water approximation to the solution of (A 1) in the forms

$$\Phi = \Phi_0(\boldsymbol{x}) + \Phi_1(\boldsymbol{x})\, z \tag{A 4a}$$

and $$\boldsymbol{A} = \boldsymbol{z}_1 \times [\boldsymbol{\Psi}_0(\boldsymbol{x}) \cosh(z/\delta) + \delta \boldsymbol{\Psi}_1(x) \sinh(z/\delta)] \quad (\boldsymbol{z}_1 \cdot \boldsymbol{\Psi}_{0,1} \equiv 0), \tag{A 4b}$$

where $\boldsymbol{x} \equiv (x, y)$ and $\boldsymbol{z}_1$ is the unit vector in the z-direction. The corresponding approximations to the complex amplitude of $\boldsymbol{U}$ and W are

$$\boldsymbol{U} = \nabla\Phi_0 - \delta^{-1}\boldsymbol{\Psi}_0 \sinh(z/\delta) - \boldsymbol{\Psi}_1 \cosh(z/\delta) \tag{A 5a}$$

and $$W = -z\nabla^2\Phi_0 + \Phi_1 + (\nabla\cdot\boldsymbol{\Psi}_0)\cosh(z/\delta) + \delta(\nabla\cdot\boldsymbol{\Psi}_1)\sinh(z/\delta). \tag{A 5b}$$

Substituting (A 4a) and (A 5a, b) into (A 2) and (A 3) and invoking (1.1), we obtain

$$\Phi_0 = (g/l\omega)\,\mathscr{L}Z, \quad \Phi_1 = -\mathrm{i}\omega Z + \nabla\cdot\{[z_0 - \delta D^{-1}(\cosh \hbar - 1 + \lambda_{\mathrm{b}} \sinh \hbar)]\,\nabla\Phi_0\}, \tag{A 6a, b}$$

$$\boldsymbol{\Psi}_0 = \delta D^{-1}(\cosh \hbar_{\mathrm{b}} + \lambda_{\mathrm{b}} \sinh \hbar_{\mathrm{b}} - \cosh \hbar_{\mathrm{f}} - \lambda_{\mathrm{f}} \sinh \hbar_{\mathrm{f}})\,\nabla\Phi_0, \tag{A 6c}$$

$$\boldsymbol{\Psi}_1 = D^{-1}(\sinh \hbar_{\mathrm{b}} + \lambda_{\mathrm{b}} \cosh \hbar_{\mathrm{b}} + \sinh \hbar_{\mathrm{f}} + \lambda_{\mathrm{f}} \cosh \hbar_{\mathrm{f}})\,\nabla\Phi_0, \tag{A 6d}$$

and (3.5), where $\hbar$, λ_{b} and λ_{f} are defined by (3.7),

$$\hbar_{\mathrm{b}} = \frac{h}{\delta}, \quad \hbar_{\mathrm{f}} = \frac{z_0}{\delta}, \quad D = (1 + \lambda_{\mathrm{b}}\lambda_{\mathrm{f}}) \cosh \hbar + (\lambda_{\mathrm{b}} + \lambda_{\mathrm{f}}) \sinh \hbar. \tag{A 7a–c}$$

The complex amplitude of the volumetric flux implied by (A 5), (A 6) and (3.7) is

$$\int_{-h}^{z_0} \boldsymbol{U}\,\mathrm{d}z = H\nabla\Phi_0 = (g/\mathrm{i}\omega)\, H\nabla\mathscr{L}Z. \tag{A 8}$$

Appendix B. Solution of (4.10)

Invoking (4.6b), we rewrite (4.10a) in the form

$$E'' - E = [(1 - C)\,E']' \equiv (qE')'. \tag{B 1}$$

It follows from (2.3a), (2.5b) and (4.5d) that $q = O(\psi^2) = O(\psi_{\mathrm{c}}^2\,\mathrm{e}^{-2\xi})$ as $\xi \to \infty$. Solving (B 1) by variation of parameters, invoking (4.10b), and integrating by parts, we obtain

$$E(\xi) = c\,\mathrm{e}^{-\xi} + \int_\xi^\infty [q(\eta)\,E'(\eta)]' \sinh(\eta - \xi)\,\mathrm{d}\eta = c\,\mathrm{e}^{-\xi} - \int_\xi^\infty q(\eta)\,E'(\eta) \cosh(\eta - \xi)\,\mathrm{d}\eta, \tag{B 2}$$

where the constant c is determined by (4.10c).

The integral equation (B 2) may be solved by iteration (cf. Lighthill 1957), starting from $E = \mathrm{e}^{-\xi}$. Substituting this first approximation into (B 2) and invoking (4.10c), we obtain the second approximations

$$E(\xi) = \mathrm{e}^{-\xi}\left\{1 + \int_\xi^\infty [1 + \mathrm{e}^{-2(\eta-\xi)}]\,q(\eta)\,\mathrm{d}\eta - \tfrac{1}{2}\int_0^\infty (1 - \mathrm{e}^{-2\eta})\,q(\eta)\,\mathrm{d}\eta\right\}, \tag{B 3}$$

$$E_0 = 1 + \int_0^\infty \mathrm{e}^{-2\xi} q(\xi)\,\mathrm{d}\xi, \quad E_0' = -1 - q_0, \tag{B 4a, b}$$

where $$q_0 \equiv q(0) = 1 - \cos^3 \psi_c \tag{B 5}$$

and $O(q_0^2)$ errors are implicit. The approximation $q = q_0 \mathrm{e}^{-2\xi}$, which gives the correct value of $q(0)$ and the correct asymptotic behaviour of q, reduces (B 4*a*) to $E_0 = 1 + \frac{1}{4}q_0$.

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